Journal of Mathematical Analysis and Applications **258,** 1–12 (2001)
doi:10.1006/jmaa.2000.7058, available online at http://www.idealibrary.com on IDEAL®

Global Existence and Stability of Solutions of Matrix Riccati Equations

Jonq Juang[1]

Submitted by Horst R. Thieme

Received February 16, 1999

We consider a matrix Riccati equation containing two parameters c and α. The quantity c denotes the average total number of particles emerging from a collision, which is assumed to be conservative (i.e., $0 < c \leq 1$), and $\alpha (0 \leq \alpha < 1)$ is an angular shift. Let $S = \{(c, \alpha): 0 < c \leq 1$ and $0 \leq \alpha < 1\}$. Stability analysis for two steady-state solutions $X_{\min}$ and $X_{\max}$ are provided. In particular, we prove that $X_{\min}$ is locally asymptotically stable for $S - \{(1, 0)\}$, while $X_{\max}$ is unstable for $S - \{(1, 0)\}$. For $c = 1$ and $\alpha = 0$, $X_{\min} = X_{\max}$ is neutral stable. We also show that such equations have a global positive solution for $(c, \alpha) \in S$, provided that the initial value is small and positive. © 2001 Academic Press

I. INTRODUCTION

This paper is concerned with the global existence and stability problem of the matrix Riccati equation of the form

$$X' = B - AX - XD + XCX := \mathscr{F}(X), \tag{1a}$$

$$X(0) = X_0. \tag{1b}$$

Here, A, B, C, and D are matrices with the structure

$$\begin{aligned} A = {} & \operatorname{diag}\left[\frac{1}{c\omega_1(1+\alpha)}, \frac{1}{c\omega_2(1+\alpha)}, \ldots, \frac{1}{c\omega_n(1+\alpha)}\right] \\ & - \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \left[\frac{c_1}{2\omega_1}, \frac{c_2}{2\omega_2}, \ldots, \frac{c_n}{2\omega_n}\right] \\ := {} & \operatorname{diag}[\delta_1, \delta_2, \ldots, \delta_n] - eq^T := D_1 - eq^T, \end{aligned} \tag{2}$$

[1] Supported in part by NSC of R.O.C., Taiwan. E-mail: jjuang@math.nctu.edu.tw.

0022-247X/01 $35.00

$$D = \operatorname{diag}\left[\frac{1}{c\omega_1(1-\alpha)}, \frac{1}{c\omega_2(1-\alpha)}, \ldots, \frac{1}{c\omega_n(1-\alpha)}\right] - qe^T \quad (3)$$

$$:= \operatorname{diag}[d_1, d_2, \ldots, d_n] - qe^T := D_2 - qe^T, \quad (4)$$

$$B = ee^T, \quad (5)$$

$$C = qq^T, \quad (6)$$

and the initial matrix X_0 is nonnegative, i.e., $(X_0)_{ij} \geq 0$ for all i, j. Equation (1a) contains two parameters, c an α. The quantity c denotes the average total number of particles emerging from a collision, which is assumed to be conservative (i.e., $0 \leq c \leq 1$), and $\alpha (0 \leq \alpha < 1)$ is an angular shift. The data $\{\omega_i\}_{i=1}^n$ and $\{c_i\}_{i=1}^n$ are sets of the Gauss–Legendre nodes and weights, respectively, on $[0, 1]$ with

$$1 > \omega_1 > \omega_2 > \cdots > \omega_n > 0,$$

and

$$\sum_{i=1}^{n} c_i = 1, \qquad c_i > 0, \qquad i = 1, 2, \ldots, n.$$

Such an equation is induced via invariant imbedding [2–5, 10, 11]), and the integration formula from an "angularly shifted" transport model [6, 7] in the slab geometry.

The solutions of (1) exhibit interesting behavior with increasing slab thickness. The equation, with slab thickness z as a parameter, can be analyzed in the context of a dynamical equation.

The purpose of this paper is twofold. First, stability analysis of Eq. (1) for two steady-state solutions $X_{\min}$ and $X_{\max}$ is provided. In particular, we show that the steady state $X_{\min}$ is locally asymptotically stable for all $0 < c \leq 1$ and $0 \leq \alpha < 1$, except that $c = 1$ and $\alpha = 0$, while $X_{\max}$ is unstable for such c and α. For $c = 1$ and $\alpha = 0$, $X_{\min}(= X_{\max})$ is neutral stable. Second, we show that Eq. (1) has a global positive solution for all $0 < c \leq 1$ and $0 \leq \alpha < 1$. In Section 2, we recorded some of the needed results concerning the steady-state solutions of Eq. (1). The main results are given in Sections 3 and 4.

II. STEADY-STATE SOLUTIONS

In the terminology of dynamical equations, the steady-state solutions to (1) satisfy

$$B - AX - XD + XCX = 0. \quad (7)$$

Let the matrix H be defined in block form by

$$H := \begin{bmatrix} D & -C \\ B & -A \end{bmatrix}; \tag{8}$$

we shall call this matrix a Hamiltonian-like matrix of Eq. (1). The complete solution bifurcation diagram of Eq. (2) has recently been obtained in [8] by considering the invariant subspace of H. Some of the results needed to study the stability of (1) are recorded in the following:

THEOREM 2.1 (Lemma 2.1 of [8]). *The matrix H, as defined in* (8), *has only real eigenvalues* $\{-\mu_n, \dots, -\mu_1, \lambda_1, \dots, \lambda_n\}$, *which are arranged in an ascending order. Those eigenvalues of H satisfy the following secular equation $f(\lambda)$ of $H - \lambda I$*:

$$f(\lambda) = 1 - \sum_{i=1}^{n} \frac{q_i}{d_i - \lambda} - \sum_{i=1}^{n} \frac{q_i}{\delta_i + \lambda}.$$

Moreover, the following assertions and estimates hold:

(i) Let δ_i and d_i, $i = 1, 2, \dots, n$, be given as (2) and (4), respectively. Then

$$-\delta_n < -\mu_n < -\delta_{n-1} < \cdots < -\delta_2 < -\mu_2 < -\delta_1 < -\mu_1 \le 0,$$
$$0 \le \lambda_1 < d_1 < \lambda_2 < d_2 < \cdots < \lambda_n < d_n.$$

(ii) $\mu_1 = 0$ only if $c = 1$.

(iii) $\lambda_1 = 0$ only if $c = 1$ and $\alpha = 0$.

(iv) For $\alpha = 0$, $\mu_i = \lambda_i$, $i = 1, 2, \dots, n$.

THEOREM 2.2 (Theorems 3.3 and 3.4 of [8]). *Let $0 < c \le 1$ and $0 \le \alpha < 1$. Equation* (2) *has a unique nonnegative solution for $c = 1$ and $\alpha = 0$. Otherwise, it has two nonnegative solutions, say $X_{\min}$ and $X_{\max}$ with $X_{\max} \ge X_{\min} > 0$. Moreover, the spectrum $\sigma(D - CX_{\min})$ of $D - CX_{\min}$ is $\{\lambda_1, \lambda_2, \dots, \lambda_n\}$, and that of $D - CX_{\max}$ is $\{-\mu_1, \lambda_2, \dots, \lambda_n\}$.*

THEOREM 2.3 (see Theorem 5.4 of [8]). *The minimum solutions $X_{\min}$ of Eq.* (2) *are strictly increasing in c (for fixed α) and decreasing in α (for fixed c).*

III. LINEARIZED STABILITY

Considering the linearized operator of $\mathscr{F}$ at $X = X_*$, where X_* is a stationary solution of (1a), we have that

$$\mathscr{F}'(X_*)R = -AR - RD + RCX_* + X_*CR. \tag{9}$$

The eigenvalue problems of $\mathscr{F}'(X_*)$ can then be formulated as

$$\mathscr{F}'(X_*)R = \lambda R. \tag{10}$$

or, equivalently,

$$-(A - X_* C)R - R(D - CX_*) = \lambda R. \tag{11}$$

To see the eigenvalues of $A - X_* C$, we need the following lemmas. Set

$$a_* = \frac{1}{2}\sum_{j=1}^{n} c_j \sum_{k=1}^{n} \frac{c_k}{\omega_k}(X_*)_{kj}, \qquad b_* = \frac{1}{2}\sum_{i=1}^{n} c_i \sum_{k=1}^{n} \frac{c_k}{\omega_k}(X_*)_{ik},$$

and

$$\alpha_* = \frac{1}{2}\sum_{j=1}^{n} c_j \omega_j \sum_{k=1}^{n} \frac{c_k}{\omega_k}(X_*)_{kj}, \qquad \beta_* = \frac{1}{2}\sum_{i=1}^{n} c_i \omega_i \sum_{k=1}^{n} \frac{c_k}{\omega_k}(X_*)_{ik}.$$

Here $* =$ min or max.

LEMMA 3.1. (i) *If* $c = 1$ *and* $\alpha = 0$, *then* $a_{\min} = b_{\min} = a_{\max} = b_{\max} = 1$. (ii) *If* $c = 1$, *and* $\alpha \neq 0$, *then* $a_{\min} < \frac{1+\alpha}{1-\alpha}$, $b_{\min} = \frac{1-\alpha}{1+\alpha}$, $a_{\max} = \frac{1+\alpha}{1-\alpha}$, *and* $b_{\max} > \frac{1-\alpha}{1+\alpha}$. (iii) *For all* c *and* $\alpha \neq 0$, $b_{\max} > \frac{1-\alpha}{1+\alpha}$.

Proof. Consider the component form of (7). We get that

$$\left(\frac{1}{\omega_i(1+\alpha)} + \frac{1}{\omega_j(1-\alpha)}\right)X_{ij} = c\left(1 + \frac{1}{2}\sum_{k=1}^{n}\frac{c_k}{\omega_k}X_{ik}\right)\left(1 + \frac{1}{2}\sum_{k=1}^{n}\frac{c_k}{\omega_k}X_{kj}\right). \tag{12}$$

Multiplying Eq. (12) by $c_i c_j$ and summing the resulting equation, we have

$$\frac{a_*}{1+\alpha} + \frac{b_*}{1-\alpha} = \frac{c}{2}(1 + a_*)(1 + b_*). \tag{13}$$

The first assertion of the lemma now follows from (13) and the fact that for $c = 1$ and $\alpha = 0$, $X_{\min} = X_{\max}$. After some algebra, (13) reduces to

$$[(1-\alpha)a_* - (1+\alpha)][(1+\alpha)b_* - (1-\alpha)] = (1-c)(1-\alpha^2)(a_* + 1)(b_* + 1). \tag{14}$$

Noting that the right-hand side of (14) is nonnegative we thus conclude, via the fact that $X_{\max} \geq X_{\min}$, that

$$a_{\min} \leq \frac{1+\alpha}{1-\alpha} \qquad \text{and} \qquad b_{\min} \leq \frac{1-\alpha}{1+\alpha}, \tag{15a}$$

and

$$a_{\max} \geq \frac{1+\alpha}{1-\alpha} \qquad \text{and} \qquad b_{\max} \geq \frac{1-\alpha}{1+\alpha}. \tag{15b}$$

Note also, via (14), that for $c = 1$, we have

$$\alpha^* = \frac{1+\alpha}{1-\alpha} \qquad \text{or} \qquad b_* = \frac{1-\alpha}{1+\alpha}.$$

We next show that for $c = 1$ and $\alpha \neq 0$, it is impossible to have both $a_* = \frac{1+\alpha}{1-\alpha}$ and $b_* = \frac{1-\alpha}{1+\alpha}$. To see this, multiplying (12) by $c_i c_j \omega_i$, and $c_i c_j \omega_j$, respectively, and summing the resulting equations, we get, respectively,

$$\frac{2\beta_*}{1-\alpha} + \frac{1}{1+\alpha}\sum_{i=1}^{n}\sum_{j=1}^{n} c_i c_j X_{ij} = \left[\frac{1}{2} + \beta_* + \frac{a_*}{2} + \beta_* a_*\right], \tag{16a}$$

and

$$\frac{2\alpha_*}{1+\alpha} + \frac{1}{1-\alpha}\sum_{i=1}^{n}\sum_{j=1}^{n} c_i c_j x_{ij} = \left[\frac{1}{2} + \alpha_* + \frac{b_*}{2} + \alpha_* b_*\right]. \tag{16b}$$

We have used the property of Gauss–Legendre nodes and weights, i.e., $\sum_{i=1}^{n} c_i \omega_i = \int_0^1 \omega \, d\omega = \frac{1}{2}$, to justify (16). Now, multiplying (16a) and (16b), respectively, by $(1+\alpha)$ and $(1-\alpha)$, and taking the difference of the resulting equations, we get

$$\begin{aligned} 2\left(\frac{1+\alpha}{1-\alpha}\beta_* - \frac{1-\alpha}{1+\alpha}\alpha_*\right) \\ &= \alpha + (1+\alpha)\beta_* - (1-\alpha)\alpha_* + \left(\frac{1+\alpha}{2}\right)a_* - \left(\frac{1+\alpha}{2}\right)b_0 \\ &\quad + (1+\alpha)a_*\beta_* - (1-\alpha)\alpha_* b_*. \end{aligned} \tag{17}$$

If $a_* = \frac{1+\alpha}{1-\alpha}$ and $b_* = \frac{1-\alpha}{1+\alpha}$, Eq. (17) would then yield that

$$\gamma := \frac{4\alpha}{(1-\alpha^2)} = 0,$$

and, hence, $\alpha = 0$, which is a contradiction. Hence, either $a_* \neq \frac{1+\alpha}{1-\alpha}$ or $b_* \neq \frac{1-\alpha}{1+\alpha}$. Moreover, for $c = 1$ and $\alpha \neq 0$, it is impossible to have

$$a_{\min} = \frac{1+\alpha}{1-\alpha}, \qquad b_{\min} < \frac{1-\alpha}{1+\alpha}. \tag{18}$$

If these were the case, substituting (18) into (17), we would have

$$\kappa := \alpha + \left(\frac{1+\alpha}{2}\right) a_{\min} - \left(\frac{1-\alpha}{2}\right) b_{\min} < 0.$$

However, this is not possible since

$$\kappa > \alpha + \frac{(1+\alpha)^2}{2(1-\alpha)} - \frac{(1-\alpha)^2}{2(1+\alpha)} = \gamma > 0.$$

We thus complete the proof of the first assertion of Lemma 3.1(ii). The second part of Lemma 3.1(ii) can be similarly obtained. If $b_{\max} = \frac{1-\alpha}{1+\alpha}$, it follows from (14) that $c = 1$. However, for $c = 1$ and $\alpha \neq 0$, $b_{\max} > \frac{1-\alpha}{1+\alpha}$. Hence, the assertion in Lemma 3.1(iii) holds as claimed. ∎

We are now ready to study the eigenvalues of $A - X_* C$.

LEMMA 3.2. (i) *The spectrum,* $\sigma(A - X_{\min} C)$, *of* $A - X_{\min} C$ *is* $\{\tilde{\mu}_1, \tilde{\mu}_2, \ldots, \tilde{\mu}_n\}$, *where* $\tilde{\mu}_i > 0$ *for* $i = 2, 3, \ldots, n$. *Moreover,* $\tilde{\mu}_1 = 0$ *at* $c = 1$; *otherwise,* $\tilde{\mu}_1 > 0$. (ii) *The spectrum* $\sigma(A - X_{\max} C)$ *of* $A - X_{\max} C$ *is* $\{\overline{\mu}_1, \overline{\mu}_2, \ldots, \overline{\mu}_n\}$, *where* $\overline{\mu}_i > 0$ *for* $i = 2, 3, \ldots, n$. *Moreover,* $\overline{\mu}_1 > 0$ (*resp.,* $= 0, < 0$) *if* $p_{c,\alpha} := \frac{2}{c(1+\alpha)} - 1 > b_{\max}$ (*resp.,* $= b_{\max}, < b_{\max}$). *In particular,* $\overline{\mu}_1 = 0$ *at* $c = 1$ *and* $\alpha = 0$, *for* $c = 1$ *and* $\alpha \neq 0$, $\overline{\mu}_1 < 0$, *and* $\overline{\mu}_1 > 0$ *if* $c(1+\alpha) < 1$.

Proof. We rewrite $A - X_* C$ as $A - X_* C = D_1 - (e + X_* q) q^T := D_1 - q_* q^T$, where $D_1 = \operatorname{diag}[\delta_1, \delta_2, \ldots, \delta_n]$ is defined as in (2). Using the Gaussian elimination technique, we see readily that δ_i, $i = 1, 2, \ldots, n$, are not eigenvalues of $A - X_* C$. Thus, for $\lambda \neq \delta_i$, $i = 1, 2, \ldots, n$, we have that

$$\begin{aligned} \det(A - X_* C - \lambda I) &= \det\left(D_1 - \lambda I - q_* q^T\right) \\ &= \det(D_1 - \lambda I) \det\left(I - (D_1 - \lambda I)^{-1} q_* q^T\right) \\ &:= \det(D_1 - \lambda I) \bar{f}(\lambda), \end{aligned}$$

where

$$\bar{f}(\lambda) = 1 - \sum_{i=1}^{n} \frac{q_i (q_*)_i}{\delta_i - \lambda}. \tag{19}$$

Hence, finding eigenvalues of $A - X_* C$ is equivalent to locating the roots of $\bar{f}(\lambda)$. Clearly, $\overline{\mu}_i$ and $\tilde{\mu}_i$, $i = 2, 3, \ldots, m$, lie between δ_{i-1} and δ_i, and, hence, are all greater than zero. To see the sign of $\overline{\mu}_1$ and $\tilde{\mu}_i$, we note that $\bar{f}(0) = 1 - \frac{c(1+\alpha)}{2}[1 + b_*]$. Clearly, for $c = 1$ and $b_* = b_{\min}$, it follows from Lemma 3.1 that $\bar{f}(0) = 0$. The last assertion of Lemma 3.2(i) follows directly from Theorem 2.3 and Lemma 3.1(ii). For $b_* = b_{\max}$,

$$\bar{f}(0) = 1 - \frac{c(1+\alpha)}{2} - \frac{c(1+\alpha) b_{\max}}{2}.$$

Hence $\bar{f}(0) < 0$ if $p_{c,\alpha} < b_{\max}$; $\bar{f}(0) = 0$ if $p_{c,\alpha} = b_{\max}$; $\bar{f}(0) > 0$ if $p_{c,\alpha} > b_{\max}$. The last assertions of Lemma 3.2(ii) follow directly from Lemma 3.1. ∎

We are now ready to state our stability results. Let

$$S = \{(c, \alpha) : 0 < c \le 1, 0 \le \alpha < 1\}.$$

THEOREM 3.3. (i) *The steady state* $X_{\min}$ *is locally asymptotically stable for* $(c, \alpha) \in S - \{(1,0)\}$ *and is neutral stable for* $c = 1$ *and* $\alpha = 0$. (ii) *The steady state* $X_{\max}$ *is unstable for* $(c, \alpha) \in S - \{(1,0)\}$.

Proof. It is well known (see, e.g., [1]) that the spectrum $\sigma(\mathscr{F}'(X_*))$ of $\mathscr{F}'(X_*)$ is equal to

$$\{-\mu - \lambda : \mu \in \sigma(A - X_* C) \text{ and } \lambda \in \sigma(D - C X_*)\}. \tag{20}$$

Now, the first assertion of Theorem 3.3 follows from Lemma 3.2(i) and Theorems 2.1 and 2.2. To complete the proof, it then suffices to show that for $(c, \alpha) \in S - \{(1,0)\}$,

$$\mu_1 > \overline{\mu}_1 \qquad \text{if } \overline{\mu}_1 \ge 0. \tag{21}$$

To this end, let $g(\lambda) = f(-\lambda)$, where $f(\lambda)$ is given as in Theorem (2.1), and define

$$h(\lambda) := \bar{f}(\lambda) - g(\lambda) = \sum_{i=1}^{n} \frac{q_i}{d_i + \lambda} - \sum_{i=1}^{n} \frac{q_i[(q_{\max})_i - 1]}{\delta_i - \lambda}.$$

For $0 \leq \lambda < \delta_1$,

$$h'(\lambda) = -\sum_{i=1}^{n} \frac{q_i}{(d_i + \lambda)^2} - \sum_{i=1}^{n} \frac{q_i[(q_{\max})_i - 1]}{(\delta_i - \lambda)^2} < 0,$$

and $h(0) = \frac{c}{2}(1 - \alpha) - \frac{c}{2}(1 + \alpha)b_{\max} < 0$. We have used Lemma 3.1(iii) to justify the last inequality. Therefore, $\bar{f}(\lambda) < g(\lambda)$ for all $0 \leq \lambda < \delta_1$. Hence, (21) holds as claimed. We thus complete the proof of the theorem. ∎

IV. GLOBAL EXISTENCE

Our objective in this section is to investigate the global solution of Eq. (1). We note that the local version of the main result, Theorem 4.2, in this section is a direct consequence of Theorems 9.1 and 9.2 of Reid [9]. To study the global solution of Eq. (1), we first rewrite (1) as an equivalent integral formulation. To this end, we begin with writing Eq. (1) as

$$X' + D_1 X + XD_2 = B + eq^T X + Xqe^T + XCX. \tag{22}$$

Premultiplying and postmultiplying Eq. (22) by the integration factors $e^{-(z-s)D_1}$ and $e^{-(z-s)D_2}$, respectively, and integrating the resulting equation with respect to s from 0 to z, we obtain

$$\begin{aligned} X(z) &= e^{-zD_1} X_0 e^{-zD_2} \\ &\quad + \int_0^z e^{-(z-s)D_1} \big[B + eq^T X(s) + X(s) qe^T \\ &\qquad\qquad + X(s) C X(s) \big] e^{-(z-s)D_2} \, ds \\ &:= (WX)(z) := e^{-zD_1} X_0 e^{-zD_2} + \sum_{i=1}^{4} (W_i X)(z), \end{aligned} \tag{23}$$

where the operators W_i, $i = 1, 2, 3, 4$, are defined as

$$(W_1 X)(z) = \int_0^z e^{-(z-s)D_1} B e^{-(z-s)D_2} \, ds,$$

$$(W_2 X)(z) = \int_0^z e^{-(z-s)D_1} eq^T X(s) e^{-(z-s)D_2} \, ds,$$

$$(W_3 X)(z) = \int_0^z e^{-(z-s)D_1} X(s) qe^T e^{-(z-s)D_2} \, ds,$$

and

$$(W_4 X)(z) = \int_0^z e^{-(z-s)D_1} X(s) C X(s) e^{-(z-s)D_2} \, ds.$$

Let us define the standard Picard iteration $\{X^{(m)}(z)\}$ by

$$X^{(0)}(z) = 0 \tag{24a}$$

$$X^{(m+1)}(z) = WX^{(m)}(z). \tag{24b}$$

Notation: Let $A = (a_{ij})$ and $B = (b_{ij})$ be two square matrices of the same size; we shall write $A \geq B$ if $a_{ij} \geq b_{i,j}$ for all i, j.

LEMMA 4.1. *If $X_{\min} \geq X_0 \geq 0$, then W is a monotone operator and $0 \leq X^{(m)}(z) \leq X^{(m+1)}(z) \leq X_{\min}$ for all $z \in [0, \infty)$ and all $m \in N$. Moreover, $X^{(m)}$ is nondecreasing in z provided that $B - D_1 X_0 - X_0 D_2 \geq 0$.*

Proof. It is clear that W is a monotone operator provided $X_0 \geq 0$. The first two inequalities are a direct consequence of an induction. To see the last inequality, assuming that $X^{(m-1)}(z) \leq X_{\min}$ for all z, we have that

$$\begin{aligned}
X^{(m)}(z) &\leq e^{-zD_1} X_0 e^{-zD_2} \\
&\qquad + \int_0^z e^{-(z-s)D_1} \big[B + eq^T X_{\min} + X_{\min} q e^T \\
&\qquad\qquad + X_{\min} C X_{\min} \big] e^{-(z-s)D_2}\, ds \\
&= e^{-zD_1} X_0 e^{-zD_2} + \int_0^z e^{-(z-s)D_1} [D_1 X_{\min} + X_{\min} D_2] e^{-(z-s)D_2}\, ds \\
&= X_{\min} - e^{-zD_1} (X_{\min} - X_0) e^{-zD_2} \\
&\leq X_{\min}.
\end{aligned}$$

To complete the proof, we assume that $X^{(m-1)}(z)$ is nondecreasing in z. Set

$$\begin{aligned}
K^{(m-1)}(z) &= B + eq^T X^{(m-1)}(z) + X^{(m-1)}(z) q e^T \\
&\qquad + X^{(m-1)}(z) C X^{(m-1)}(z),
\end{aligned}$$

and, hence, $K^{(m-1)}(z)$ is increasing. Differentiating $WX^{(m-1)}(z)$ with respect to z, one obtains that

$$\begin{aligned}
&\frac{d}{dz} WX^{(m-1)}(z) \\
&\quad = -e^{-zD_1} (D_1 X_0 + X_0 D_2) e^{-zD_2} + K^{(m-1)}(z) \\
&\qquad - \int_0^z e^{-(z-s)D_1} \big[D_1 K^{(m-1)}(s) + K^{(m-1)}(s) D_2 \big] e^{-(z-s)D_2}\, ds \\
&\quad \geq -e^{-zD_1} (D_1 X_0 + X_0 D_2) e^{-zD_2} + e^{-zD_1} K^{(m-1)}(z) e^{-zD_2} \\
&\quad = e^{-zD_1} (B - D_1 X_0 - X_0 D_2) e^{-zD_2} \geq 0.
\end{aligned}$$

The fact that $K^{(m-1)}(s)$ are increasing in s has been used to justify the first inequality above. ∎

THEOREM 4.2. (i) *Let* $0 < c \le 1$, *and let* $0 \le \alpha < 1$. *Moreover, the initial value* X_0 *is so small that* $X_{\min} \ge X_0 \ge 0$, $B - D_1 X_0 - X_0 D_2 \ge 0$. *Then the sequence* $X^{(m)}(z)$ *converges pointwise to a continuous function* $X^{(\infty)}(z)$ *on* $[0, \infty)$. (ii) $X^{(\infty)}(z)$ *is an nondecreasing function* (*in* z) *on* $[0, \infty)$, *which is a global solution of* (1). (iii) *The limit of* $X^{(\infty)}(z)$ *as* $z \to \infty$ *exists, say* $X^{(\infty)}$. (iv) *Moreover, the limit* $X^{(\infty)}$ *is a solution of steady-state Eq.* (7). *Furthermore,* $X^{(\infty)} = X_{\min}$.

Proof. The assertions of Theorem 4.2(i), (ii), and (iii) follow from the Monotone Convergence Theorem and Lemma 4.1. To complete the proof of the last assertion of the theorem, we need to show that X_∞ is a solution of (7), or, equivalently, X_∞ satisfies

$$\begin{aligned} X_\infty &= \int_0^\infty e^{-sD_1}\left[B + eq^T X_\infty + X_\infty q e^T + X_\infty C X_\infty\right] e^{-sD_2}\, ds \\ &:= \sum_{i=1}^{n} \lim_{z\to\infty} \overline{W}_i(z) X_\infty, \end{aligned}$$

where

$$\overline{W}_1(z) X_\infty = \int_0^z e^{-sD_1} B e^{-sD_2}\, ds,$$

$$\overline{W}_2(z) X_\infty = \int_0^z e^{-sD_1} eq^T X_\infty e^{-sD_2}\, ds,$$

$$\overline{W}_3(z) X_\infty = \int_0^z e^{-sD_1} X_\infty q e^T e^{-sD_2}\, ds,$$

and

$$\overline{W}_4(z) X_\infty = \int_0^z e^{-sD_1} X_\infty C X_\infty e^{-sD_2}\, ds.$$

To this end, we need to show that $\lim_{z\to\infty}[(\overline{W}_i X)(z) X_\infty - (W_2 X)(z)] = 0$, $i = 1, 2, 3, 4$. Here W_i are defined in (24a). We illustrate only $i = 2, 4$; the other limits can be similarly obtained. Now,

$$\overline{W}_2(z) X_\infty - (W_2 X)(z) = \int_0^z e^{-sD_1} eq^T (X_\infty - X(z-s)) e^{-sD_2}\, ds.$$

Dividing the integration interval $[0, z]$ into two parts, we have the following estimates:

$$\begin{aligned}\int_0^{z/2} & e^{-sD_1} eq^T (X_\infty - X(z-s)) e^{-sD_2}\, ds \\ &\leq \int_0^{z/2} e^{-sD_1} eq^T \left(X_\infty - X\left(\tfrac{z}{2}\right)\right) e^{-sD_2}\, ds \\ &\leq \int_0^{z/2} e^{-sD_1} D_1 D_1^{-1} eq^T \left(X_\infty - X\left(\tfrac{z}{2}\right)\right) ds \\ &\leq D_1^{-1} eq^T \left(X_\infty - X\left(\tfrac{z}{2}\right)\right).\end{aligned}$$

Furthermore,

$$\begin{aligned}\int_{z/2}^{z} & e^{-sD_1} eq^T (X_\infty - X(z-s)) e^{-sD_2}\, ds \\ &\leq \int_{z/2}^{z} e^{-sD_1} D_1 D_1^{-1} eq^T (X_\infty - X_0)\, ds \\ &\leq e^{-(z/2)D_1} D_1^{-1} q^T (X_\infty - X_0).\end{aligned}$$

The fact that $X(z)$ is increasing in z has been used to justify the above inequalities. We now turn to the next estimate:

$$\begin{aligned}\overline{W}_4(z) X_\infty & - \left(\overline{W}_4 X\right)(z) \\ &= \int_0^{z} e^{-sD_1} \left(X_\infty C X_\infty - X(z-s) C X(z-s)\right) e^{-sD_2}\, ds.\end{aligned}$$

We have, via similar estimates, that

$$\begin{aligned}\int_0^{z/2} & e^{-sD_1} \left(X_\infty C X_\infty - X(z-s) C X(z-s)\right) e^{-sD_2}\, ds \\ &\leq D_1^{-1} \left(X_\infty C X_\infty - X\left(\tfrac{z}{2}\right) C X\left(\tfrac{z}{2}\right)\right),\end{aligned}$$

and that

$$\begin{aligned}\int_{z/2}^{z} & e^{-sD_1} \left(X_\infty C X_\infty - X(z-s) C X(z-s)\right) e^{-sD_2}\, ds \\ &\leq e^{-(z/2)D_1} D_1^{-1} (X_\infty C X_\infty - X_0 C X_0).\end{aligned}$$

Therefore, $(\overline{W}_i X)(z) - \overline{W}_i(z) X_\infty$ can be made arbitrarily small by choosing z sufficiently large. Hence X_∞ satisfies Eq. (2) as claimed. Finally, X_∞ must be equal to $X_{\min}$. Since $0 \leq X^{(n)}(z) \leq X_{\min}$ for all n and z, and $X_\infty \leq X_{\min}$, then $X_\infty = X_{\min}$. ∎

REFERENCES

1. R. Bellman, "Introduction to Matrix Analysis," 2nd ed., McGraw-Hill, New York, 1970.
2. R. Bellman, R. E. Kalaba, and C. Prestrud, "Invariant Imbedding and Radiative Transfer in Slabs of Finite Thickness," Elsevier, New York, 1963.
3. R. Bellman and G. M. Wing, "An Introduction to Invariant Imbedding," Wiley, New York, 1975.
4. I. W. Busbridge, "The Mathematics of Radiative Transfer," Cambridge Univ. Press, London/New York, 1960.
5. S. Chandrasekhar, "Radiative Transfer," Dover, New York, 1960.
6. F. Coron, Computation of the asymptotic states for linear half space kinetic problem, *Transport. Theory Statist. Phys.* **19**, No. 2 (1990), 89–114.
7. B. D. Ganapol, An investigation of a simple transport model, *Transport Theory Statist. Phys.* **21**, Nos. 1 and 2 (1992), 1–37.
8. J. Juang and W. W. Lin, Nonsymmetric algebraic Riccati equations and Hamiltonian-like matrices, *SIAM J. Matrix Anal. Appl.* **20**, No. 1 (1998), 228–243.
9. W. T. Reid, "Riccati Differential Equations," Academic Press, New York, 1972.
10. A. Shimizu and K. Aoki, "Application of Invariant Embedding to Reactor Physics," Academic Press, New York, 1972.
11. G. M. Wing, "An Introduction to Transport Theory," Wiley, New York, 1962.